Schrödinger equation revisited

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The time-dependent Schrödinger equation is a cornerstone of quantum physics and governs all phenomena of the microscopic world. However, despite its importance, its origin is still not widely appreciated and properly understood. We obtain the Schrödinger equation from a mathematical identity by a slight generalization of the formulation of classical statistical mechanics based on the Hamilton–Jacobi equation. This approach brings out most clearly the fact that the linearity of quantum mechanics is intimately connected to the strong coupling between the amplitude and phase of a quantum wave.

The birth of the time-dependent Schrödinger equation was perhaps not unlike the birth of a river. Often, it is difficult to locate uniquely its spring despite the fact that signs may officially mark its beginning. Usually, many bubbling brooks and streams merge suddenly to form a mighty river. In the case of quantum mechanics, there are so many convincing experimental results that many of the major textbooks do not really motivate the subject. Instead, they often simply postulate the classical-to-quantum rules as

$$E \to i\hbar \frac{\partial}{\partial t} \quad \text{and} \quad \boldsymbol{p} \to \frac{\hbar}{i} \boldsymbol{\nabla}$$

for the energy E and momentum p, where $\hbar$ is Planck's constant divided by 2π and operators are understood as acting on the wave function $\psi = \psi(\boldsymbol{r}, t)$. The reason given is that "it works."

For example, the Schrödinger equation is then obtained (1, 2) from the classical Hamiltonian $H \equiv p^2/(2m) + V$ for a particle of mass m in a potential $V = V(\boldsymbol{r}, t)$ as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \psi + V\psi. \qquad [1]$$

This approach is unfortunate. Many of us recall feeling dissatisfied with this recipe.

It was the left-hand side of Eq. 1 that was the sticking point for Schrödinger (3–7). Wave equations usually involved second time derivatives. It is thus somewhat ironic that classical mechanics à la the Hamilton–Jacobi equation yield a nonlinear wave equation that is similar to Eq. 1, namely,

$$i\hbar \frac{\partial \psi^{(cl)}}{\partial t} = -\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \psi^{(cl)} + V\psi^{(cl)} + Q\left[\psi^{(cl)}\right]\psi^{(cl)}. \qquad [2]$$

Here, Q is a nonlinear potential that depends on $\psi^{(cl)}$ (Eq. 14). The basis for Eq. 2 is the fact that energy and momentum are obtained from the Hamilton–Jacobi theory by taking time and space derivatives of the action, as we discuss in the following.

There are, of course, many ways (8–15) in which to obtain the time-dependent Schrödinger equation, with the most prominent being the one developed by Feynman (14) based on the path integral. In this article, we obtain the Schrödinger equation (Eq. 1) and discuss the connection with Eq. 2 in three steps:

i) Starting from the nonlinear wave equation (Eq. 2), we assume that we search for a wave equation for a scalar wave containing only a first-order derivative in time and a second-order derivative in space, and we establish a mathematical identity involving derivatives of a complex-valued field.

ii) A description (16–20) of classical statistical mechanics in terms of a classical matter wave whose phase is given by the classical action and is governed by the Hamilton–Jacobi equation (21), and whose amplitude is defined by the square root of the Van Vleck determinant (22) and satisfies a continuity equation (16–18, 22, 23), leads via our mathematical identity to a nonlinear wave equation.

iii) However, a linear wave equation (i.e., the Schrödinger equation) emerges from our mathematical identity when we couple amplitude and phase in a democratic way (i.e., the phase determines the dynamics of the amplitude, and vice versa).

It is this mutual coupling between amplitude and phase that defines a quantum matter wave and ensures the linearity of the wave equation. Indeed, in the classical matter wave, this coupling is broken; the phase still determines, via the continuity equation, the dynamics of the amplitude, but the equation of motion of the phase (i.e., the Hamilton–Jacobi equation) is independent of the amplitude. It is for this reason that the wave equation is nonlinear.

Our article is organized as follows: we first establish the mathematical identity and then turn to the nonlinear wave equation corresponding to a classical matter wave; the next section is dedicated to the task of arriving at the linear Schrödinger equation; and we conclude by summarizing our results and providing an outlook.

To focus on the essential ideas, we have moved detailed calculations or introductory material to an appendix. In the first section (Appendix), we use standard relations of vector calculus to obtain the building blocks of our mathematical identity. In the second section (Appendix), we provide more insight into the Van Vleck determinant by considering a 1D description.

Mathematical Identity

In this section, we formulate the problem and spell out our assumptions about the wave equation. We then obtain the mathematical identity, which is at the heart of our approach toward the Schrödinger equation.

Our goal is to obtain a wave equation for matter. For this purpose, we consider the most elementary case of a scalar particle described by a single complex-valued function

$$Z = Z(\boldsymbol{r}, t) \equiv A(\boldsymbol{r}, t) e^{i\theta(\boldsymbol{r}, t)}, \qquad [3]$$

consisting of the real-valued and positive amplitude $A = A(\boldsymbol{r}, t)$ and the real-valued phase $\theta = \theta(\boldsymbol{r}, t)$. Here, $\boldsymbol{r}$ combines the Cartesian coordinates of the particle in three space dimensions and t is the coordinate time.

At this point, the use of a complex, rather than a real, valued wave is by no means obvious. However, this choice will become clear in the next section when we discuss classical matter waves. Indeed, our treatment will show that we use a complex function as a very efficient shorthand notation to combine two coupled real-valued equations into a single equation.

Wave equations contain derivatives with respect to the space and time coordinates. For example, the familiar wave equation for an electromagnetic wave involves second-order derivatives with respect to space and time. However, we now consider the more elementary case of a wave equation with only a first-order derivative with respect to time but second-order derivatives with respect to space.

In *Appendix*, we establish the identities

$$i\frac{\partial Z}{\partial t} = \left(\frac{i}{2A^2}\frac{\partial}{\partial t}A^2 - \frac{\partial \theta}{\partial t}\right)Z \qquad [4]$$

and

$$\nabla^2 Z = \left[\frac{i}{A^2}\nabla \cdot (A^2\nabla\theta) - (\nabla\theta)^2 + \frac{\nabla^2 A}{A}\right]Z, \qquad [5]$$

which follow when we complete the differentiations on the right-hand sides of these equations, together with the representation (Eq. **3**) of the complex-valued function Z in terms of the amplitude A and the phase θ.

Because the first-order time derivative and the second-order space derivative given by the Laplacian of Z have different dimensions, that is, [time]$^{-1}$ vs. [space]$^{-2}$, we need to introduce a constant β when we add the two derivatives. In this way, we arrive at the mathematical identity

$$i\frac{\partial Z}{\partial t} + \beta\nabla^2 Z = \left\{i\frac{1}{2A^2}\left[\frac{\partial}{\partial t}A^2 + 2\beta\nabla \cdot (A^2\nabla\theta)\right] \right.$$
$$\left. + \left[-\frac{\partial \theta}{\partial t} - \beta(\nabla\theta)^2 + \beta\frac{\nabla^2 A}{A}\right]\right\}Z. \qquad [6]$$

Because A and θ are real, the expressions in the two square brackets on the right-hand side of Eq. **6** are also real. Moreover, they bear a great similarity to a continuity equation and the Hamilton–Jacobi equation of classical mechanics, respectively. In the next two sections, we shall use physical arguments to identify the constant β, simplify the right-hand side of Eq. **6**, and obtain in this way the two wave equations corresponding to classical and quantum mechanics.

Classical Matter Waves

So far, we have only used mathematics. To make contact with physics, we recall in the present section the Hamilton–Jacobi equation (21). We then define a classical matter wave whose phase is given by the classical action and whose amplitude follows from an appropriate combination of second derivatives of the action, that is, from the Van Vleck determinant. Finally, the mathematical identity will show that a so-defined classical matter wave satisfies a nonlinear wave equation.

Classical Mechanics as Field Theory. Central to the present section is the Hamilton–Jacobi equation (21)

$$-\frac{\partial S^{(cl)}}{\partial t} = \frac{(\nabla S^{(cl)})^2}{2m} + V \qquad [7]$$

for a nonrelativistic classical particle of mass m moving in a potential $V = V(\mathbf{r}, t)$, which may even be time-dependent. Here,

$S^{(cl)} \equiv S^{(cl)}(\mathbf{r}, \boldsymbol{\alpha}, t)$ denotes the classical action and the vector $\boldsymbol{\alpha}$ combines three constants of the motion.

Indeed, the Hamilton–Jacobi equation (Eq. **7**) is a partial differential equation of the first order in the three Cartesian coordinates x_k with $k = 1, 2,$ and 3 and time. Hence, we expect its solution to depend on $3 + 1 = 4$ independent constants α_l of integration. Because only derivatives of $S^{(cl)}$ enter into the differential equation, we can always add a constant α_0 to any solution of $S^{(cl)}$ and obtain another solution. When we disregard this trivial constant of integration, we arrive at only three, that is, one for each coordinate. Obviously, in the case of N coordinates, we find N constants α_l of integration.

Most relevant for the present problem of obtaining a wave equation is the fact that Eq. **7** implies (16–18, 22, 23) the Van Vleck continuity equation

$$\frac{\partial}{\partial t}D + \nabla \cdot \left[D\frac{\nabla S^{(cl)}}{m}\right] = 0, \qquad [8]$$

where

$$D \equiv \left|\frac{\partial^2 S^{(cl)}}{\partial x_k \partial \alpha_l}\right| \qquad [9]$$

denotes the Van Vleck determinant.

In *Appendix*, we motivate this conservation law by deriving it for a 1D motion. For the more complicated case of N coordinates and N constants of integration, we refer to other sources (16–18, 22, 23).

Nonlinear Wave Equation. Next, we note that the expression in the first square bracket on the right-hand side of the mathematical identity (Eq. **6**) is identical to the left-hand side of the Van Vleck continuity equation (Eq. **8**), provided $D \equiv A^2$ and $2\beta\theta \equiv S^{(cl)}/m$. This feature, together with the fact that the classical action $S^{(cl)}$ defines wave fronts, suggests consideration (16–18) of a wave

$$\psi^{(cl)} \equiv A^{(cl)}e^{i\theta^{(cl)}} \equiv D^{1/2}\exp\left[\frac{i}{\hbar}S^{(cl)}\right] \qquad [10]$$

whose amplitude $A^{(cl)} \equiv D^{1/2}$ is the square root of the Van Vleck determinant and whose phase $\theta^{(cl)} \equiv S^{(cl)}/\hbar$ is the classical action $S^{(cl)}$ divided by a constant $\hbar$ with the dimension of an action. This constant ensures that the phase of the wave is dimensionless. Needless to say, there is no justification to identify this constant with the reduced Planck's constant. Indeed, we could have chosen any quantity with the dimension of an action to make the phase dimensionless. Nevertheless, our analysis brings out the critical role of this constant in the transition from classical to quantum mechanics.

When we substitute the wave ansatz in Eq. **10** into the mathematical identity (Eq. **6**), we arrive at

$$\left\{i\frac{\hbar}{2D}\left[\frac{\partial D}{\partial t} + \frac{2\beta}{\hbar}\nabla \cdot \left(D\nabla S^{(cl)}\right)\right]\right.$$
$$+ \left[-\frac{\partial S^{(cl)}}{\partial t} - \frac{\beta}{\hbar}\left(\nabla S^{(cl)}\right)^2 + \hbar\beta\frac{\nabla^2|\psi^{(cl)}|}{|\psi^{(cl)}|}\right]\right\}\psi^{(cl)} \qquad [11]$$
$$= i\hbar\frac{\partial \psi^{(cl)}}{\partial t} + \hbar\beta\nabla^2\psi^{(cl)}.$$

The choice

$$\beta \equiv \frac{\hbar}{2m} \qquad [12]$$

for the free parameter β allows us now to take advantage of the Van Vleck continuity equation (Eq. **8**), and the first square bracket in Eq. **11** vanishes. Moreover, due to the value of β given by Eq. **12**, the Hamilton–Jacobi equation (Eq. **7**) replaces the first two terms in the second square bracket in Eq. **11** by the potential V, and we find (16–20) for $\psi^{(\mathrm{cl})}$ the wave equation

$$i\hbar\frac{\partial \psi^{(\mathrm{cl})}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^{(\mathrm{cl})} + V\psi^{(\mathrm{cl})} + Q\left[\psi^{(\mathrm{cl})}\right]\psi^{(\mathrm{cl})} \qquad [13]$$

with

$$Q\left[\psi^{(\mathrm{cl})}\right] \equiv \frac{\hbar^2}{2m}\frac{\nabla^2\left|\psi^{(\mathrm{cl})}\right|}{\left|\psi^{(\mathrm{cl})}\right|}. \qquad [14]$$

As a result, the wave equation (Eq. **13**) for $\psi^{(\mathrm{cl})}$ (i.e., for a classical matter wave) is very close to the Schrödinger equation of quantum mechanics but is nonlinear. The nonlinearity is due to the quantity Q, which involves the ratio of the Laplacian of the amplitude $|\psi^{(\mathrm{cl})}|$ of the classical wave and $|\psi^{(\mathrm{cl})}|$. Moreover, it is proportional to the square of $\hbar$. Despite the appearance of $\hbar$ in this nonlinear wave equation, we are still in classical mechanics. Indeed, it is exactly the potential Q that enforces the classicality of Eq. **13**; that is, it ensures that the two basic equations of this wave description of classical mechanics, the Hamilton–Jacobi equation for $S^{(\mathrm{cl})}$ and the Van Vleck continuity equation for the density $|\psi^{(\mathrm{cl})}|^2 = D$, are free of $\hbar$. For this reason, we shall refer to this potential as the classicality-enforcing potential.

We also note that Q will play an important role in the quantum description, where it appears in the dynamical equation for the action and carries the name Madelung–Bohm quantum potential (24–26). However, in this equation, it enters with the opposite sign, as we shall show in the next section.

In this approach, we have used the classical Hamilton–Jacobi equation (Eq. **7**), together with the Van Vleck continuity equation (Eq. **8**), to obtain from the mathematical identity (Eq. **6**) the nonlinear wave equation (Eq. **13**). Needless to say, it is also possible to move in the opposite direction. Indeed, the nonlinear wave equation implies the two equations we have started from.

Probability Interpretation and the Imaginary Unit. The analysis leading to Eq. **13** teaches us three important lessons: (*i*) The classical action $S^{(\mathrm{cl})}$ as defined by the Hamilton–Jacobi equation (Eq. **7**) represents a major portion (1, 2) of the phase of a quantum wave; (*ii*) the amplitude $A^{(\mathrm{cl})} \equiv D^{1/2}$ of the wave $\psi^{(\mathrm{cl})}$ given by Eq. **9** depends on the phase $S^{(\mathrm{cl})}/\hbar$, but the Hamilton–Jacobi equation (Eq. **7**) determining $S^{(\mathrm{cl})}$ is independent of D; and (*iii*) the Van Vleck continuity equation (Eq. **8**), together with the wave representation (Eq. **10**), provides us with the identification of the classical density $\rho^{(\mathrm{cl})} \equiv D \equiv |\psi^{(\mathrm{cl})}|^2$ and the classical current $j^{(\mathrm{cl})} \equiv \rho v \equiv D(\nabla S^{(\mathrm{cl})}/m)$.

The last feature yields the interpretation (1, 2) of $|\psi^{(\mathrm{cl})}|^2$ as a density. Indeed, we have already reached a statistical theory, but of classical mechanics only. In this framework, we consider an ensemble of particles moving along classical trajectories in phase space given by Newton's equation. The nonlinear wave equation (Eq. **13**) for $\psi^{(\mathrm{cl})}$ is completely equivalent to classical statistical mechanics. Because Eq. **13** is complex, it contains two real-valued equations, namely, the Hamilton–Jacobi equation (Eq. **7**) for $S^{(\mathrm{cl})}$ and the Van Vleck continuity equation (Eq. **8**) for D.

In the past, Stückelberg (27), Wheeler (28), and many others have addressed the question of why the imaginary unit appears so prominently in quantum mechanics but not in standard formulations of classical mechanics. However, the complex-valued function $\psi^{(\mathrm{cl})}$, which obeys the nonlinear Schrödinger equation (Eq. **13**), demonstrates that the appearance of the imaginary unit i is

not a characteristic feature of quantum mechanics but, rather, reflects the fact that the underlying dynamics rest on two equations rather than one: the continuity equation and the Hamilton–Jacobi equation. At this point, it is of no importance that the latter implies the former. Therefore, complex numbers are just a useful tool to combine two real equations into a single complex equation.

We conclude our discussion of classical matter waves by noting that the appearance of i in quantum mechanics as a mathematical convenience rather than a necessity is also confirmed by the formulation in terms of the Wigner phase space distribution function (29). Indeed, this quantity is always real. We will return to this point in *Conclusions and Outlook*.

Quantum Matter Waves

So far, we have achieved a wave description (16–20) of classical statistical mechanics, and the corresponding wave equation is nonlinear. In this section, we obtain by a special choice of the mutual coupling between the amplitude and phase of the wave a linear wave equation. Because the wave equation of the so-defined wave that follows from our mathematical identity is the Schrödinger equation, we refer to this type of wave as a quantum matter wave.

Linear Wave Equation. To make the transition from the nonlinear classical wave equation to the linear Schrödinger equation, that is, from classical to quantum physics, we first note that due to the nonlinearity, Eq. **13** does not allow standing waves, that is, a superposition of a right-going wave and a left-going wave. For this purpose, waves of the type

$$\psi^{(\mathrm{q})} \equiv A^{(\mathrm{q})}\exp\left(\frac{\mathrm{i}}{\hbar}S^{(\mathrm{q})}\right) \qquad [15]$$

must satisfy a *linear* wave equation that follows from the mathematical identity (Eq. **6**) when we absorb the classicality-enforcing potential Q defined by Eq. **14**, which is the last term of the second square bracket in the mathematical identity (Eq. **6**) in the action $S^{(\mathrm{q})}$.

Indeed, with $\theta = S^{(\mathrm{q})}/\hbar$ and the choice $\beta = \hbar/(2m)$ given by Eq. **12**, the mathematical identity (Eq. **6**) takes the form

$$\begin{aligned} i\hbar\frac{\partial \psi^{(\mathrm{q})}}{\partial t} &+ \frac{\hbar^2}{2m}\nabla^2\psi^{(\mathrm{q})} \\ &= \left\{ \frac{\mathrm{i}}{2\left(A^{(\mathrm{q})}\right)^2}\left[\frac{\partial}{\partial t}\left(A^{(\mathrm{q})}\right)^2 + \nabla\cdot\left(\left(A^{(\mathrm{q})}\right)^2\frac{\nabla S^{(\mathrm{q})}}{m}\right)\right] \right. \\ &\quad \left. + \left[-\frac{\partial S^{(\mathrm{q})}}{\partial t} - \frac{1}{2m}\left(\nabla S^{(\mathrm{q})}\right)^2 + Q\left[A^{(\mathrm{q})}\right]\right]\right\}\psi^{(\mathrm{q})}, \end{aligned} \qquad [16]$$

where we have recalled the definition (Eq. **14**) of the potential Q.

Motivated by the fact that classical matter waves satisfy the Van Vleck continuity equation (Eq. **8**), we now postulate that a similar equation should also hold true for quantum matter waves; that is, we assume the relation

$$\frac{\partial}{\partial t}\left(A^{(\mathrm{q})}\right)^2 + \nabla\cdot\left(\left(A^{(\mathrm{q})}\right)^2\frac{\nabla S^{(\mathrm{q})}}{m}\right) = 0. \qquad [17]$$

As a consequence, Eq. **16** reduces to

$$i\hbar\frac{\partial \psi^{(q)}}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi^{(\mathrm{q})} = \left(-\frac{\partial S^{(\mathrm{q})}}{\partial t} - \frac{1}{2m}\left(\nabla S^{(\mathrm{q})}\right)^2 + Q\left[A^{(q)}\right]\right)\psi^{(\mathrm{q})}. \qquad [18]$$

So far, we have not specified the equation of motion of the quantum action $S^{(\mathrm{q})}$. In the case of the classical action $S^{(\mathrm{cl})}$, the Hamilton–Jacobi equation (Eq. **7**) plays this role and brings

the potential V into the wave equation for $\psi^{(\text{cl})}$. Moreover, it leads to the nonlinear wave equation (Eq. **13**) because the classicality-enforcing potential $Q[A^{(\text{cl})}]$ that is already presented in the mathematical identity (Eq. **6**) is not eliminated. Because our goal is to achieve a linear wave equation, we can attach $Q[A^{(\text{q})}]$ to the dynamical equation of the quantum action $S^{(\text{q})}$ and postulate the equation of motion:

$$-\frac{\partial S^{(q)}}{\partial t} = \frac{1}{2m}\left(\boldsymbol{\nabla}S^{(q)}\right)^2 + V - Q\left[A^{(q)}\right]. \quad \textbf{[19]}$$

With the help of this definition of the dynamics of $S^{(\text{q})}$, we arrive at the familiar Schrödinger equation:

$$i\hbar\frac{\partial \psi^{(q)}}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi^{(q)} + V\psi^{(q)}. \quad \textbf{[20]}$$

We emphasize that we have achieved this linear wave equation by removing the classicality-enforcing potential Q from the wave and associating it with the action. Indeed, in comparison to the classical Hamilton–Jacobi equation (Eq. **7**) governing $S^{(\text{cl})}$, the dynamics of the quantum action $S^{(\text{q})}$ given by Eq. **19** depend on the classicality-enforcing potential Q. In contrast to the nonlinear wave equation (Eq. **13**), where Q enters with a positive sign, it appears in Eq. **19** with a minus sign. More importantly, the system of Eqs. **17** and **19**, consisting of the quantum continuity and the quantum Hamilton–Jacobi equations, contains Planck's constant explicitly. Hence, the potential $-Q[A^{(\text{q})}]$ in the equation of motion for the quantum action $S^{(\text{q})}$ plays a similar role as $+Q[A^{(\text{cl})}]$ does in the nonlinear wave equation for $\psi^{(\text{cl})}$. In Eq. **19**, it defines the quantum nature of the equation for $S^{(\text{q})}$ in the same way as $Q[A^{(\text{cl})}]$ determines the classical nature of the nonlinear equation. It is interesting that $\hbar$ appears in the nonlinear wave equation despite the fact that it is of classical nature. This property stands out most clearly in the classical Hamilton–Jacobi equation (Eq. **7**) and in the continuity equation (Eq. **8**), which are independent of $\hbar$. In contrast, in the corresponding quantum equations, it does not drop out. Nevertheless, for $\hbar \to 0$, the quantum potential $-Q[A^{(\text{q})}]$ vanishes and the quantum Hamilton–Jacobi equation reduces to the classical equation, in complete agreement with the correspondence principle.

Difference Between Classical and Quantum. Eqs. **17** and **19** are motivated by three principles: (*i*) conservation of matter as suggested by the Van Vleck continuity equation (Eq. **8**) of classical mechanics, (*ii*) an appropriate generalization of the classical Hamilton–Jacobi equation (Eq. **7**) dictated by the goal to achieve a linear wave equation, and (*iii*) a smooth classical-quantum transition in accordance with the correspondence principle. Needless to say, Eqs. **17** and **19** also follow from the Schrödinger equation (Eq. **20**) and the decomposition equation (Eq. **15**) into amplitude and phase, which is the path taken by Madelung (24) in 1926 and Bohm (25, 26) in 1952.

Although Eqs. **17** and **19** seem to differ only slightly from their classical counterparts, Eqs. **8** and **7**, there is a dramatic conceptual difference. Indeed, the phase $S^{(\text{q})}$ influences through its gradient in the continuity equation (Eq. **17**) the amplitude $A^{(\text{q})}$ of the quantum wave $\psi^{(\text{q})}$. In turn, $A^{(\text{q})}$ enters through the potential $-Q[A^{(\text{q})}]$ appearing in the quantum Hamilton–Jacobi equation (Eq. **19**) into the phase $S^{(\text{q})}$. This coupling scheme between the amplitude and phase of a quantum wave is in sharp contrast to the corresponding coupling scheme of a classical wave. Here, the classical action $S^{(\text{cl})}$ determines the amplitude $A^{(\text{cl})} \equiv D^{1/2}$ through the definition (Eq. **9**) of the Van Vleck determinant. However, because $D \equiv (A^{(\text{cl})})^2$ does not appear in the classical Hamilton–Jacobi equation (Eq. **7**), the classical action $S^{(\text{cl})}$ is independent of $A^{(\text{cl})}$.

Conclusions and Outlook

In summary, we have obtained the Schrödinger equation starting from a mathematical identity (Eq. **6**). In our argument, the Hamilton–Jacobi equation of classical mechanics has played a key role. Indeed, it not only suggests a specific choice of the parameter β in Eq. **6** connecting space and time derivatives, but it shows us that we can linearize the nonlinear wave equation when we include the potential Q in the equation of motion for the action rather than in the equation of motion for the wave. At the same time, we require a continuity equation similar to the one of Van Vleck describing classical statistical mechanics. In this way, the amplitude of a quantum wave depends on its phase, and vice versa. This rather symmetrical dependence is broken in classical mechanics, giving rise to the nonlinear wave equation.

Our analysis also shines some light on the old question: Why the imaginary unit? The complex wave function, which depends on the position variable and time as a parameter, is just an efficient way of combining two coupled real equations. The square of the amplitude of the wave determines the probability density, and the derivative of the phase yields the momentum. In contrast, the Wigner function (29) is always real and lives in phase space. Therefore, it depends on the position as well as the momentum variable. In this sense the two degrees of freedom of the complex-valued wave function, amplitude and phase, are connected to the real-valued Wigner function of the two variables position and momentum.

It is also remarkable that the potential V enters the scene only through the Hamilton–Jacobi equation. Needless to say, we could have also added on both sides of the mathematical identity the term γVZ with another constant γ to match the different dimensions. In this case, we would have obtained an expression on the left-hand side of Eq. **6**, which is even closer to the Schrödinger equation.

However, one might then wonder why not add on both sides a nonlinear function of the amplitude of the wave, such as $\kappa A^2 Z$, and arrive at a nonlinear Schrödinger equation of the Gross–Pitaevskii type. The reason is because it is the classical Hamilton–Jacobi equation that brings in V, which does not contain such terms to begin with. They do not emerge in quantum theory either as confirmed by landmark experiments (30–32) verifying that the quantum mechanics of noninteracting particles are linear.

In the same vein, we could add to the first-order time derivative given in Eq. **4** a first-order space derivative, such as $\boldsymbol{\nabla}Z$, rather than second-order space derivatives. However, in this case, we not only have the problem of the different dimensions of the two derivatives but the fact that they are of different vectorial natures. Indeed, the time derivative is a scalar, whereas the space derivative involves a gradient and is thus a vector. Therefore, this addition of the two derivatives will require another vector. Moreover, we might want to move from scalar waves to vector waves. We suspect that one might also get some deeper insight into the Weyl equation or the Dirac equation in this way.

In this article, we have only addressed the Schrödinger equation corresponding to a single nonrelativistic particle without internal degrees of freedom. Thus, we have not touched the phenomenon of entanglement, which Schrödinger called *the* trait of quantum mechanics. Entanglement arises from the interaction of several particles and manifests itself in a single wave function that depends on the coordinates of all particles but is not separable. This feature suggests that entanglement is best described in configuration rather than in phase space. The connection between entanglement and Born's rule has been illuminated by Zurek (33), and it would be fascinating to build a bridge between our approach and that of Zurek (33).

Unfortunately, this question, as well as the derivation of the Weyl or Dirac equation, is beyond the scope of the present article and will be the topic of a future publication.

Appendix

Operator Identities. In this section, we provide more details about the derivation and building blocks of the mathematical identity (Eq. 6). Here, we consider differentiations of the complex-valued field

$$Z(\boldsymbol{r},t) \equiv A(\boldsymbol{r},t)e^{i\theta(\boldsymbol{r},t)}, \qquad [21]$$

defined by its real-valued and positive amplitude $A = A(\boldsymbol{r}, t)$ and real-valued phase $\theta = \theta(\boldsymbol{r}, t)$. In particular, we consider the first-order derivative with respect to time and the second-order derivative with respect to space. For this purpose, it is convenient to use Eq. 21 to express A in terms of Z, that is,

$$A = Ze^{-i\theta}. \qquad [22]$$

We start our discussion by establishing the relation

$$i\frac{\partial Z}{\partial t} = \left(\frac{i}{2A^2}\frac{\partial}{\partial t}A^2 - \frac{\partial\theta}{\partial t}\right)Z \qquad [23]$$

for the time derivative of Z by noting the identity

$$\frac{\partial}{\partial t}A^2 = 2A\left(\frac{\partial Z}{\partial t}e^{-i\theta} - iZe^{-i\theta}\frac{\partial\theta}{\partial t}\right) = 2A^2\left(\frac{\partial Z}{\partial t}\frac{1}{Z} - i\frac{\partial\theta}{\partial t}\right), \qquad [24]$$

which follows directly from Eq. 22.

Next, we verify the formula

$$\boldsymbol{\nabla}^2 Z = \left[\frac{i}{A^2}\boldsymbol{\nabla}\cdot\left(A^2\boldsymbol{\nabla}\theta\right) - (\boldsymbol{\nabla}\theta)^2 + \frac{\boldsymbol{\nabla}^2 A}{A}\right]Z, \qquad [25]$$

which involves second derivatives with respect to position.

For this purpose, we first note that

$$\boldsymbol{\nabla}\cdot\left(A^2\boldsymbol{\nabla}\theta\right) = 2A\boldsymbol{\nabla}A\cdot\boldsymbol{\nabla}\theta + A^2\boldsymbol{\nabla}^2\theta, \qquad [26]$$

which leads with the identity

$$\boldsymbol{\nabla}A = \boldsymbol{\nabla}Ze^{-i\theta} - iA\boldsymbol{\nabla}\theta \qquad [27]$$

following from Eq. 22 to

$$\boldsymbol{\nabla}\cdot\left(A^2\boldsymbol{\nabla}\theta\right) = 2A\boldsymbol{\nabla}Z\cdot\boldsymbol{\nabla}\theta e^{-i\theta} - 2iA^2(\boldsymbol{\nabla}\theta)^2 + A^2\boldsymbol{\nabla}^2\theta. \qquad [28]$$

Moreover, we find from Eq. 27 the relation

$$\boldsymbol{\nabla}^2 A = \boldsymbol{\nabla}^2 Ze^{-i\theta} - 2i\boldsymbol{\nabla}Z\cdot\boldsymbol{\nabla}\theta e^{-i\theta} - A(\boldsymbol{\nabla}\theta)^2 - iA\boldsymbol{\nabla}^2\theta. \qquad [29]$$

When we substitute Eqs. 28 and 29 into the right-hand side of Eq. 25, we indeed arrive on the left-hand side.

Van Vleck Continuity Equation. In this section, we motivate the Van Vleck continuity equation Eq. 8 by considering the case of a single particle of mass m moving along the x axis. Here, the Van Vleck determinant reduces to

$$D \equiv \frac{\partial^2 S^{(cl)}}{\partial x\partial\alpha} \equiv \frac{\partial^2 S^{(cl)}}{\partial\alpha\partial x}, \qquad [30]$$

where $S^{(cl)} = S^{(cl)}(x,\ \alpha,\ t)$ follows from the Hamilton–Jacobi equation

$$-\frac{\partial S^{(cl)}}{\partial t} = \frac{1}{2m}\left(\frac{\partial S^{(cl)}}{\partial x}\right)^2 + V. \qquad [31]$$

When we differentiate D as defined by Eq. 30, interchange the order of differentiations, and take advantage of the Hamilton–Jacobi equation (Eq. 31), we find

$$\frac{\partial D}{\partial t} = \frac{\partial^2}{\partial x\partial\alpha}\left(\frac{\partial S^{(cl)}}{\partial t}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial\alpha}\left[\frac{1}{2m}\left(\frac{\partial S^{(cl)}}{\partial x}\right)^2 + V\right]\right). \qquad [32]$$

Because the potential V does not depend on the constant α of integration, we obtain

$$\frac{\partial D}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{m}\frac{\partial S^{(cl)}}{\partial x}\frac{\partial^2 S^{(cl)}}{\partial\alpha\partial x}\right), \qquad [33]$$

that is, the Van Vleck continuity equation

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial x}\left(D\frac{1}{m}\frac{\partial S^{(cl)}}{\partial x}\right) = 0 \qquad [34]$$

for one dimension. Here, we have recalled the definition in Eq. 30 of D.

We emphasize that the corresponding derivation for N space coordinates x_k and N constants α_l of integration is slightly more complicated and needs as an additional ingredient the Jacobi equation (34)

$$d\mathcal{M} = \mathcal{M}\text{Tr}\left(\mathbf{M}^{-1}d\mathbf{M}\right) \qquad [35]$$

for the differential of a determinant $\mathcal{M}$ of a matrix $\mathbf{M}$ in terms of the trace Tr of the product $\mathbf{M}^{-1}d\mathbf{M}$ consisting of the inverse $\mathbf{M}^{-1}$ and the differential $d\mathbf{M}$ of $\mathbf{M}$. For the details of this derivation, we refer the reader to other sources (16–18, 23).

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